Section 3.3: Reduced echelon form

New vocabulary:

- reduced echelon matrix
- Homogeneous system of equations
- Trivial solution (to a HSOE) = zero solution
- Principal diagonal $\begin{bmatrix} \cdot & & ? \\ & \cdot & \\ ? & & \cdot \end{bmatrix}$ = leading diagonal
- Identity matrix $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}$
- Gauss-Jordan elimination

the right hand side is all 0

echelon matrix: $\begin{bmatrix} 0 & 0 & * & * & * & \cdots & * \\ 0 & 0 & 0 & 0 & * & \cdots & * \\ & & & \vdots & & & \\ 0 & - & & - & - & - & 0 \end{bmatrix}$

1. Zero rows at the bottom
2. Each leading entry occurs in a later column than the leading entries of earlier rows.

reduced echelon matrix has 1. & 2.

and

3. Leading entries are 1, and all other entries in a column with l.e. are 0.

$$\begin{bmatrix} 0 & 0 & 1 & * & 0 & * & & * \\ 0 & 0 & 0 & 0 & 1 & * & & * \\ & & & \vdots & & & & \\ 0 & & & - & & - & - & 0 \end{bmatrix}$$

Recall: leading entries, free variables

Question:

Use Gauss-Jordan elimination to solve

$$\begin{aligned} 2y + 4z &= 8 \\ 2x + 4y + z &= -1 \\ x + 3y + 2z &= 3 \end{aligned}$$

① ↔ ③ $\quad \begin{aligned} x + 3y + 2z &= 3 \\ 2x + 4y + z &= -1 \\ 2y + 4z &= 8 \end{aligned}$

② → ② − 2① $\quad \begin{aligned} x + 3y + 2z &= 3 \\ 0 - 2y - 3z &= -7 \\ 2y + 4z &= 8 \end{aligned}$

③ → ③ + ② $\quad \begin{aligned} x + 3y + 2z &= 3 \\ 0 - 2y - 3z &= -7 \\ z &= 1 \end{aligned}$

② → $-\frac{1}{2}$② $\quad \begin{aligned} x + 3y + 2z &= 3 \\ y + \tfrac{3}{2}z &= \tfrac{7}{2} \\ z &= 1 \end{aligned}$

① → ① − 2③, ② → ② − $\frac{3}{2}$③ $\quad \begin{aligned} x + 3y &= 1 \\ y &= 2 \\ z &= 1 \end{aligned}$

Question:

Find the reduced echelon form of $\begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

and of $\begin{bmatrix} 0 & 2 & 4 & 8 \\ 2 & 4 & 1 & -1 \\ 1 & 3 & 2 & 3 \end{bmatrix}$

Solution: ① ↔ ③ ...

to get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

① → ① − 3② $\quad \begin{aligned} x &= -5 \\ y &= 2 \\ z &= 1 \end{aligned}$

Solution: $(x, y, z) = (-5, 2, 1)$

Theorems

- Reduced echelon form is unique. (Theorem 1)
- More variables than equations imply the number of solutions is either 0 or infinity. (Theorem 3+)
- A homogeneous system always has a solution.
- A system with n variables and n equations has a unique solution if and only if the reduced echelon form to the left of the line is the identity.
- If there are infinitely many solutions, then there are free variables ✓

A homogeneous system with n variables and n equations has a unique solution if and only if the coefficient matrix has reduced echelon form the identity. (Theorem 4)

n variables

m

m equations

If $n > m$, there is always a free variable

If the system is consistent, we get infinitely many solutions, inconsistent: no solutions,

The zero solution = the trivial solution

The identity on the left produces a unique solution ✓.

If we don't get the identity on the left there is a row of zeros and a free variable.

Question:

Find the reduced echelon form of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 2 \\ 3 & 6 & 7 \end{bmatrix}$

a. $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix}$ leading entry must be 1

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ✓

d. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e. None of the above

$2 \to -\frac{1}{8}\, 2$ $\qquad \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$1 \to 1 - 5\, 2$ $\qquad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

True or false?

1. If a system of linear equations has more equations than unknown variables then there is no solution.

2. If a system has fewer equations than variables there is always a solution.

3. If a system has fewer equations than variables and there is at least one solution then there are infinitely many solutions.